# Structure Invariance for Uncertain Nonlinear Systems

R. Castro-Linares and C. H. Moog

*Abstract*—**A unified study of three control problems associated to multi-input multi-output nonlinear systems with uncertainties is presented, namely, input-state linearization by static state feedback, input–output decoupling by static state feedback and input–output decoupling by dynamic compensation. For each of these problems, geometric conditions which describe intrinsic structural invariance properties are given.**

## I. INTRODUCTION

During the last two decades, nonlinear control theory was very actively developed (see e.g., [14], [16], [20] and the references therein). In particular, geometric and algebraic methods have received considerable attention in the literature. Exact control problems such as linearization and decoupling of the closed-loop system have been studied in various ways, e.g., linearization of the closed-loop state equations [13], [15], linearization of the closed-loop input–output map [14], and input–output decoupling of the closed-loop system [9], [12].

A major question arises when the model of the system contains uncertain elements such as constant or varying parameters that are not known or imperfectly known. Under such imperfect knowledge of the model, one tries to design a control such that the model can still achieve the desired closed-loop behavior.

The matter of robustness of such control methods is often argued in some case studies. The idea for studying some classes of uncertainties in the system model is growing in the current literature since it aims to develop a general theory for robustness. To deal with uncertain nonlinear systems, two main approaches have been proposed in the literature: adaptive control and Lyapunov-based control. The first one is applied to systems with parameterized uncertainties (see e.g., [22]), while the second allows nonparameterized uncertainties. The Lyapunov-based approach relies on an explicit construction of a Lyapunov function from which a state feedback control is synthesized assuming bounds on the uncertainties. To obtain either stabilization or tracking, however, some assumptions were introduced regarding the structure of the uncertainties. That is to say that the uncertainties have to enter into the state equation in a certain way; such conditions are often referred to as matching conditions.

Some studies have been carried out on the stability analysis of uncertain dynamical systems not satisfying matching conditions. In [21], it is pointed out that mismatched uncertainty may affect robust stability when the feedback linearizing method is applied. Such systems which contain mismatched uncertainties are not considered in the rest of this paper.

In addition, several authors have contributed to the robust tracking problem for nonlinear systems with uncertainties. A multivariable tracking problem is studied in [10], using a measurement of the tracking error which is a general function of the system's state and input; the resulting controller is robust in the sense that the tracking error is ultimately bounded in the presence of modeling errors which satisfy the matching conditions. A similar result is obtained in the work of [1] where the so-called generalized matching condition for a class of nonlinear systems is introduced. More recently, the robust output tracking problem was addressed in [18] for a class of single-input single-output (SISO) nonlinear systems, of which the uncertainties may not satisfy the conventional matching condition, using a Lyapunov-based approach.

In this note, we consider a multi-input multi-output (MIMO) nonlinear system in the presence of uncertainties, referred to as the uncertain system, described by

$$\Sigma^P:\ \begin{cases}\dot{x} = f(x) + \Delta f(x) + (g(x) + \Delta g(x))u,\\ y = h(x)\end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^p$ is the input vector, $y \in \mathbb{R}^p$ is the output vector, $f(x)$ and the $p$ columns $g_1(x), \cdots, g_p(x)$ of the matrix $g(x)$ are meromorphic vector fields of $x$, and the $p$ components $h_1(x), \cdots, h_p(x)$ of the vector $h(x)$ are meromorphic functions of $x$. $\Delta f(x)$ and the $p$ columns $\Delta g_1(x), \cdots, \Delta g_p(x)$ of the matrix $\Delta g(x)$ are also meromorphic vector fields of $x$ which represent the disturbance and model uncertainties.

The corresponding nonlinear system without uncertainty, called the nominal system, is then defined as

$$\Sigma:\ \begin{cases}\dot{x} = f(x) + g(x)u,\\ y = h(x)\end{cases} \tag{2}$$

i.e., $\Delta f(x) \equiv 0$ and $\Delta g(x) \equiv 0$ in (1). In the rest of the paper, dim $D$ denotes the generic dimension of a meromorphic distribution $D$, i.e., its constant dimension on a suitable dense submanifold of $\mathbb{R}^n$.

The goal of the note is to give a unified geometric study for three control problems associated to the uncertain nonlinear system $\Sigma^P$: input-state linearization by static state feedback, static state feedback decoupling, and decoupling under dynamic compensation. In particular, some geometric intrinsic conditions on the uncertainties are given for each problem. These conditions are innovative in the sense that they describe intrinsic structural invariance properties from the nominal system to the uncertain system. Under these conditions, robust trajectory tracking is studied in [3] when dynamic decoupling is considered. Equivalent results for input-state linearization and input–output decoupling by static state feedback can be found in the literature (see, for example, [1], [16], [18], [23]).

The note is organized as follows. In Section II, we recall the input-state linearization problem by static state feedback and give geometric conditions so that the uncertain system in closed-loop form has a particular representation in a new set of coordinates. The same is done in Sections III and IV for the input–output decoupling problem by static state feedback and by dynamic compensation, respectively. Finally, a conclusion is offered in Section V.

## II. INPUT-STATE LINEARIZATION BY STATIC-STATE FEEDBACK

Through this section we consider an uncertain system $\Sigma^P$ and a nominal system $\Sigma$ without outputs, i.e., each system is just defined by the corresponding state equation.

Given the vector fields $f$ and the matrix $g$, the input-state linearization problem we address here consists in supposing the existence of a state-space coordinate transformation and a regular static state feedback defined on $\mathbb{R}^n$, such that the nominal nonlinear system $\Sigma$ is equivalent to a linear controllable system

$$\dot{\xi} = A\xi + Bv \tag{3}$$

with $A$ and $B$ constant matrices of dimension $n \times n$ and $n \times m$, respectively, and $v = \mathrm{col}(v_1, \cdots, v_p)$ a new input. It is well known

Manuscript received August 9, 1993; revised December 13, 1993.

The authors are with the Laboratoire d'Automatique de Nantes, Unité Associée au C.N.R.S., Ecole Centrale de Nantes, Université de Nantes, 1 rue de la Noë, 44072 Nantes Cedex 03, France. R. Castro-Linares is on sabbatical leave from CINVESTAV-IPN, Dept. of Electrical Eng., Apdo. Postal 14-740, 07000 Mexico, D.F., Mexico.

IEEE Log Number 9403957.

0018–9286/94$04.00 © 1994 IEEE

that the sequence of distributions $D_k = \overline{g + \cdots + ad_f^k g}$, $0 \le k \le n-2$ where $\overline{D}$ denotes the involutive closure of a distribution $D$, lies in the heart of the input-state linearization problem by regular static state feedback since the cardinality $c_k$ of the controllability indexes equal to $k$ associated to the linearized system is given by [19]

$$c_k = \dim \frac{D_k^\perp}{D_{k+1}^\perp} - \dim \frac{D_{k+1}^\perp}{D_{k+2}^\perp}.$$

So the system $\Sigma$ is fully linearizable if and only if $\sum_{k=1}^{k=n} kc_k = n$. Let the coordinate transformation

$$\xi = \Phi(x) \tag{4}$$

and the regular static state feedback

$$u = \alpha(x) + \beta(x)v \tag{5}$$

with $\alpha(x)$ a $p \times 1$ vector and $\beta(x)$ a $p \times p$ invertible matrix both defined on $\mathbb{R}^n$ be a solution of the input-state linearization problem for the nominal system $\Sigma$. It is well known [14], [20] that if such a solution exists there are $p$ real-valued meromorphic functions of $x$, $\phi_i(x)$, $i = 1, \cdots, p$ which have relative degrees $r_1, \cdots, r_p$, with $r_1 + \cdots + r_p = n$, and the decoupling matrix $A(x)$ is invertible. Thus, we can set the state-space coordinate transformation (4) to be $\xi = \mathrm{col}(\xi_1, \cdots, \xi_p)$ where $\xi_i = \mathrm{col}(\xi_{i1}, \cdots, \xi_{i,r_i})$ with $\xi_{ij} = L_f^{j-1}\phi_i(x)$, $i = 1, \cdots, p$, $j = 1, \cdots, r_i$. Let us also define the sequence of distributions $D_k^P$ associated to the uncertain system $\Sigma^P$ as $D_k^P = \overline{g + \Delta g + \cdots + ad_{f+\Delta f}^k (g + \Delta g)}$, $0 \le k \le n-2$. We then have the following result.

*Proposition 2.1:* Assume that the nominal system $\Sigma$ without outputs is input-state linearizable by regular static state feedback and state diffeomorphism $\xi$. Then the uncertain system $\Sigma^P$ is input-state linearizable by regular static state feedback in the state coordinates $\xi$ with unchanged controllability indexes, if and only if

$$D_k = D_k^P, \quad \text{for } k \ge 0. \tag{6}$$

*Proof (Sufficiency):* If (6) is fulfilled, then the relative degree of any meromorphic function $\varphi(x)$ is invariant under the considered uncertainties. Moreover, the strong accessibility distribution remains unchanged. Assume that (3) is given under a Brunovsky canonical form which consists of $p$ blocks of dimension $r_i$, for $i = 1, \cdots, p$. Then, the uncertain system $\Sigma^P$ fed back with (5) takes the following form in the coordinates $\xi$

$$\dot{\xi}_i = A_i\xi_i + b_i v_i + \psi_i(\xi) + \tilde{\alpha}_i(\xi) + \tilde{\beta}_i(\xi)v, \quad i = 1, \cdots, p \tag{7}$$

with

$$A_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{8}$$

$$\psi_i(\xi) = \begin{bmatrix} \psi_{i1}(\xi) \\ \vdots \\ \psi_{i,r_i}(\xi) \end{bmatrix}, \quad \tilde{\alpha}_i(\xi) = \begin{bmatrix} 0 \\ \vdots \\ \Delta_i(\xi)\alpha(\xi) \end{bmatrix},$$

$$\tilde{\beta}_i(\xi) = \begin{bmatrix} 0 \\ \vdots \\ \Delta_i(\xi)\beta(\xi) \end{bmatrix} \tag{9}$$

where $\psi_{ij}(\xi) = L_{\Delta f} L_f^{j-1}\phi_i(x)\,|_{x=\Phi^{-1}(\xi)}$, $\Delta_i(\xi) = L_{\Delta g} L_f^{r_i - 1}\phi_i(x)\,|_{x=\Phi^{-1}(\xi)}$.

*(Necessity):* The necessity of (6) follows from the fact that the coordinates for the Brunovsky canonical form are unchanged as well as the controllability indexes.

□

*Remark 2.2:* Note that the system's property described in Proposition 2.1 is stronger than just input-state linearizability. The full characterization of the latter is not a structural problem as can be easily understood from the following example

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + \Delta f(x) + \Delta g(x)u \\ &= \begin{bmatrix} 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u + \begin{bmatrix} x_2 \\ -x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u \end{aligned}$$

where the nominal linear system switches to the uncertain system which has a different structure though input-state linearity is maintained.

*Remark 2.3:* Note that the standard matching condition [1], [4], [10] implies $\tilde{\alpha}(x) = 0$ and $\tilde{\beta}(x) = 0$, thus $\langle d\psi_{i,r_i - 1}, g + \Delta g\rangle = 0$, and $d\xi_{i1} \perp \overline{g + \Delta g + \cdots + ad_{f+\Delta f}^{r_i - 1}(g + \Delta g)}$, which is stronger than (6). The invariance of the relative degree $r_i$ has also been considered in [1] for a nonlinear system which has a single output.

The closed-loop representation (7) can be written in a more concise form as

$$\dot{\xi} = A\xi + Bv + \Psi(\xi) + \tilde{\alpha}(\xi) + \tilde{\beta}(\xi)v \tag{10}$$

with

$$A = \begin{bmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_p \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_p \end{bmatrix}, \tag{11}$$

$$\Psi(\xi) = \begin{bmatrix} \psi_1(\xi) \\ \vdots \\ \psi_p(\xi) \end{bmatrix}, \quad \tilde{\alpha}(\xi) = \begin{bmatrix} \tilde{\alpha}_1(\xi) \\ \vdots \\ \tilde{\alpha}_p(\xi) \end{bmatrix}, \quad \tilde{\beta}(\xi) = \begin{bmatrix} \tilde{\beta}_1(\xi) \\ \vdots \\ \tilde{\beta}_p(\xi) \end{bmatrix}. \tag{12}$$

The stabilization of nonlinear systems having the form (10) has been considered in the literature. For example, in [16] a Lyapunov analysis is made to study the effects of the perturbation terms which satisfy the matching conditions by means of an additional feedback control $v = \gamma(x)$. More recently, a methodology has been suggested in [23] to tackle this problem without requiring matching assumptions. There, single-input nonlinear systems without uncertainties in the input vector $g$ are studied using a combination of sliding control ideas together with the recursive construction of a closed-loop Lyapunov function.

## III. Input–Output Decoupling by Static-State Feedback

Let us now examine an uncertain system $\Sigma^P$ (1) and a nominal system $\Sigma$ (2) with outputs. We consider the existence of a regular static state feedback (5) such that the nominal feedback modified dynamics

$$\tilde{\Sigma}: \quad \begin{cases} \dot{x} = \tilde{f}(x) + \tilde{g}(x)u, \\ y = h(x) \end{cases}$$

with $\tilde{f}(x) = f(x) + g(x)\alpha(x)$ and $\tilde{g}(x) = g(x)\beta(x)$, is input–output decoupled, i.e., the $i$th input $v_i$ only affects the $i$th output $y_i$ and not the other outputs $y_j$, $j \ne i$, possibly after a relabeling of the input $v_1, \cdots, v_m$. Such a problem is known as the input–output decoupling problem by regular static state feedback.

Assume that each output of the nominal system $\Sigma$ has relative degree $r_i$, $i = 1, \cdots, p$, and that the decoupling matrix

$$A(x) = \begin{bmatrix} L_{g_1}L_f^{r_1-1}h_1(x) & \cdots & L_{g_m}L_f^{r_1-1}h_1(x) \\ \vdots & & \vdots \\ L_{g_1}L_f^{r_p-1}h_p(x) & \cdots & L_{g_m}L_f^{r_p-1}h_p(x) \end{bmatrix} \tag{13}$$

is right invertible. Then, the nominal system is input–output decouplable by a standard decoupling feedback which has the form (5) (see, for example, [14], [20]). Also, the set of functions $\xi_{ij} = L_f^{j-1}h_i(x)$, $i = 1, \cdots, p$, $j = 1, \cdots, r_i$, define a partial state-space coordinate transformation and we set, as in Section II, $\xi = \mathrm{col}(\xi_1, \cdots, \xi_p)$ with $\xi_i = \mathrm{col}(\xi_{i1}, \cdots, \xi_{i,r_i})$, and $r = r_1 + \cdots + r_p$. In addition, let $\overline{\xi}(x)$ be an $(n-r)$ vector of meromorphic functions of $x$ such that $\zeta = (\xi, \overline{\xi}) = \Phi(x)$ forms a local change of coordinates.

In the rest of the paper, the derivative of order $k$ of the output $y_i$ along the trajectories of $\Sigma$ is denoted $y_i^{(k)}$ whereas its derivative of order $k$ along the trajectories of $\Sigma^P$ is denoted $y_i^{[k]}$. Thus, we define

$$\mathcal{Y}_i^k = \mathrm{span}\{dy_i, \cdots, dy_i^{(k)}\},$$
$$\mathcal{Y}_i^{kP} = \mathrm{span}\{dy_i, \cdots, dy_i^{[k]}\}$$

for $0 \le k \le r_i - 1$ while $D^*$, $D^{*P}$ are the largest controlled invariant distributions contained in $\ker\{dh_1, \cdots, dh_p\}$ for $\Sigma$ and $\Sigma^P$, respectively. Also, $D_i^*$ and $D_i^{*P}$ denote the largest controlled invariant distributions contained in $\ker\{dh_i\}$, for the nominal and uncertain systems. Let $\tilde{\Sigma}^P$ denote the uncertain dynamics obtained after decoupling the nominal system. Then we have

*Proposition 3.1:* Suppose the nominal system $\Sigma$ is input–output decouplable by regular static state feedback. Assume

$$\mathcal{Y}_i^k = \mathcal{Y}_i^{kP}, \quad \text{for } 0 \le k \le r_i - 1,\ 1 \le i \le p \tag{14}$$

and

$$D^* = D^{*P}. \tag{15}$$

Then

i) the decoupling matrix of $\Sigma^P$ is right invertible, and
ii) with respect to the coordinates $\zeta = (\xi, \overline{\xi})$, the uncertain dynamics $\tilde{\Sigma}^P$ is

$$\tilde{\Sigma}^P: \begin{cases} \dot{\xi}_i = A_i\xi_i + b_iv_i + \psi_i(\xi, \overline{\xi}) + \tilde{\alpha}_i(\xi, \overline{\xi}) + \tilde{\beta}_i(\xi, \overline{\xi})v, \\ \dot{\overline{\xi}} = \overline{f}(\xi, \overline{\xi}) + \overline{g}(\xi, \overline{\xi}, v), \\ y_i = \xi_{i1}, \quad i = 1, \cdots, p \end{cases} \tag{16}$$

with the pairs $(A_i, b_i)$ in the Brunovsky canonical form (8), and $\psi_i$, $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ given by

$$\psi_i(\xi, \overline{\xi}) = \begin{bmatrix} \psi_{i1}(\xi_{i1}, \xi_{i2}) \\ \vdots \\ \psi_{i,r_i-1}(\xi_{i1}, \xi_{i2}, \cdots, \xi_{i,r_i}) \\ \psi_{i,r_i}(\xi, \overline{\xi}) \end{bmatrix},$$

$$\tilde{\alpha}_i(\xi, \overline{\xi}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Delta_i(\xi, \overline{\xi})\alpha(\xi, \overline{\xi}) \end{bmatrix},$$

$$\tilde{\beta}_i(\xi, \overline{\xi}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Delta_i(\xi, \overline{\xi})\beta(\xi, \overline{\xi}) \end{bmatrix}$$

where $\psi_{ij}(\xi, \overline{\xi}) = L_{\Delta f}L_f^{j-1}h_i(x) \mid_{x=\Phi^{-1}(\xi, \overline{\xi})}$ and $\Delta_i(\xi, \overline{\xi}) = L_{\Delta g}L_r^{r_i-1}h_i(x) \mid_{x=\Phi^{-1}(\xi, \overline{\xi})}$.

The conditions given in Proposition 3.1 are also necessary; these can be verified by standard computations. The special case where $\psi_{ij} = 0$, for $1 \le j \le r_i - 1$, can be found in [17]. The necessary and sufficient condition then becomes

$$\Delta f + \mathrm{span}\{\Delta g\} \in D^*.$$

*Proof of Proposition 3.1:* Since the nominal system $\Sigma$ can be decoupled, then $D^* = \cap D_i^*$. Also $\mathcal{Y}_i = D_i^{*\perp}$ and $\mathcal{Y}_i^P = D_i^{*P\perp}$. On the other hand, assumptions (14) and (15) yield

$$D^* = D^{*P} = \bigcap_{i=1}^{i=p} D_i^{*P}$$

and the system $\Sigma^P$ remains decouplable.

Since $\xi_{i1} = y_i = h_i(x)$, for $i = 1, \cdots, p$, and from the invariance of $\mathcal{Y}_i^1 = \mathrm{span}\{dy_i, d\dot{y}_i\}$ (i.e., $\mathcal{Y}_i^1 = \mathcal{Y}_i^{1P}$) one gets, for $\tilde{\Sigma}^P$

$$\dot{\xi}_{i1} = \xi_{i2} + \psi_{i1}(\xi_{i1}, \xi_{i2})$$

with $\psi_{i1} = L_{\Delta f}h_i(x)$ and $\xi_{i2} = L_fh_i(x)$. More generally, for any $j \le r_i - 1$, one has

$$\dot{\xi}_{ij} = \xi_{i,j+1} + \psi_{ij}(\xi_{i1}, \xi_{i2}, \cdots, \xi_{i,j+1})$$

where $\psi_{ij} = L_{\Delta f}L_f^{j-1}h_i(x)$ and $\xi_{i,j+1} = L_f^jh_i(x)$. The last component of $\dot{\xi}_i$ reads

$$\dot{\xi}_{i,r_i} = v_i + \psi_{i,r_i}(\xi, \overline{\xi}) + \Delta_i(\xi, \overline{\xi})\alpha(\xi, \overline{\xi}) + \Delta_i(\xi, \overline{\xi})\beta(\xi, \overline{\xi})v$$

where now

$$\psi_{i,r_i}(\xi, \overline{\xi}) = L_{\Delta f}L_f^{r_i-1}h_i(x) \mid_{x=\Phi^{-1}(\xi, \overline{\xi})}$$

and

$$\Delta_i(\xi, \overline{\xi}) = L_{\Delta g}L_f^{r_i-1}h_i(x) \mid_{x=\Phi^{-1}(\xi, \overline{\xi})}.$$

Finally, by setting

$$\overline{f}(\xi, \overline{\xi}) = L_f\overline{\xi}(x) \mid_{x=\phi^{-1}(\xi, \overline{\xi})} + L_g\overline{\xi}.\alpha(x) \mid_{x=\phi^{-1}(\xi, \overline{\xi})},$$

$$\overline{g}(\xi, \overline{\xi}, v) = L_g\overline{\xi}.\beta(x) \mid_{x=\phi^{-1}(\xi, \overline{\xi})} .v + L_{\Delta f}\overline{\xi}(x) \mid_{x=\phi^{-1}(\xi, \overline{\xi})} + L_{\Delta g}\overline{\xi}.\alpha(x) \mid_{x=\phi^{-1}(\xi, \overline{\xi})} + L_{\Delta g}\overline{\xi}.\beta(x) \mid_{x=\phi^{-1}(\xi, \overline{\xi})} .v$$

the representation (16) follows.

From (16) we can notice that the state vector $\overline{\xi}$ remains completely unobservable at the output. Thus, for the purposes of stabilization we need that $\overline{\xi}$ remain bounded for bounded $\xi$ and $v$; i.e., we require that the internal dynamics

$$\dot{\overline{\xi}} = \overline{f}(\xi, \overline{\xi}) + \overline{g}(\xi, \overline{\xi}, v)$$

be bounded-input bounded-state (BIBS) stable. This question has received considerable attention during the last few years. For instance, in [1] BIBS stability is guaranteed for an uncertain SISO nonlinear system, where $\overline{\xi}$ is chosen so that $\overline{g}(\xi, \overline{\xi}, v) \equiv 0$; the MIMO case is treated in [8] using variable structure control. The same stability aspect is considered in [18] for the SISO case but taking into account the term $\overline{g}$.

## IV. Input–Output Decoupling by Dynamic Compensation

In this section we consider again an uncertain system $\Sigma^P$ and a nominal system $\Sigma$ with outputs, i.e., systems (1) and (2). We now look at more general control loops associated with the nominal system $\Sigma$ to achieve input–output decoupling. More precisely, given the nominal system we address the problem when there exists a dynamic compensator with state $z$

$$\begin{aligned}\dot{z} &= M(x, z) + N(x, z)v\\ u &= F(x, z) + G(x, z)v\end{aligned} \tag{17}$$

where $z = \text{col}(z_1, \cdots, z_q) \in \mathbb{R}^q$, $M: \mathbb{R}^{n+q} \to \mathbb{R}^q$, $N: \mathbb{R}^{n+q} \to \mathbb{R}^{q\times p}$, $F: \mathbb{R}^{n+q} \to \mathbb{R}^p$ and $G: \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^{p\times p}$ are meromorphic mappings, and $v = \text{col}(v_1, \cdots, v_p)$ represents a new input, such that [9]

$$dy_i^{(k)} \in \text{span}\{dx, dz, dv_i, \cdots, dv_i^{(k-1)}\}, \quad i = 1, \cdots, p,\ k \geq 1 \tag{18}$$

and

$$dy_i^{(n+q)} \notin \text{span}\{dx, dz\}, \qquad i = 1, \cdots, p. \tag{19}$$

Precisely, (18) represents the noninteraction of the new inputs $v_1, \cdots, v_p$ while (19) represents the output controllability condition.

Next we address the structural invariance properties associated with the nominal and uncertain nonlinear systems under dynamic compensation. We first recall, from [5], [9], the definition of some invariant integers that are of interest to our purpose. Let us denote $y^{(k)}$ as the time derivative of order $k$ of the output $y$ along the trajectories of $\Sigma$ and $y^{[k]}$ as the same derivative along the trajectories of $\Sigma^P$. As before, the relative degree associated to each output $y_i$ is denoted by $r_i$ while $\sigma_k$ is the number of zeros at infinity of order less than or equal to $k$, $1 \leq k \leq n$. The essential order $n_{ie}$ of the output $y_i$ is defined by

$$\begin{aligned}n_{ie} = \min\{k \geq 1 \mid\ & dy_i^{(k)}\\ &\notin \text{span}\{dx, d\dot{y}, \cdots, dy^{(k-1)}, dy_{j\neq i}^{(k)}, dy^{(k+1)}, \cdots, dy^{(n)}\}\}.\end{aligned}$$

A differential $dy_i^{(k)}$, $dy_i^{[k]}$ as above is said to be essential in $\{dx, d\dot{y}, \cdots, dy^{(n)}\}$, respectively $\{dx, d\dot{y}, \cdots, dy^{[n]}\}$.

Now, we consider a nominal system $\Sigma$ which is right invertible. In this case there exist [12] a so-called Singh compensator (17) which decouples the nominal system. The decoupled nominal system $\hat{\Sigma}_d$ can be written as

$$\hat{\Sigma}_d: \begin{cases}\dot{x}_e = f_e(x_e) + g_e(x_e)v,\\ y = h(x)\end{cases}$$

where $x_e = \text{col}(x, z)$, $f_e(x_e) = \text{col}(f(x)+g(x)F(x, z), M(x, z))$ and $g_e(x_e) = \text{col}(g(x)G(x, z), N(x, z))$. Since $r_i(\hat{\Sigma}_d) = n_{ie}(\hat{\Sigma}_d)$, $i = 1, \cdots, p$ [9], $\zeta = (\xi, \hat{\xi}) = \Phi(x, z)$ defines a local change of coordinates with $\xi = \text{col}(\xi_1, \cdots, \xi_p)$ and $\xi_i = \text{col}(\xi_{i1}, \cdots, \xi_{i,n_{ie}})$ where $\xi_{ij} = L_{f_e}^{j-1}h_i(x_e)$, $i = 1, \cdots, p$, $j = 1, \cdots, n_{ie}$, $n_e = n_{1e} + \cdots + n_{pe}$, and $\hat{\xi}$ is an $(n + q - n_e)$ vector of meromorphic functions of $x$ and $z$.

Let

$$\begin{aligned}\mathcal{X} &= \text{span}\{dx\},\\ \mathcal{Y}^k &= \text{span}\{dy, \cdots, dy^{(k)}\},\\ \mathcal{Y}^{kP} &= \text{span}\{dy, \cdots, dy^{[k]}\}, \quad \text{for } k \geq 0,\\ \mathcal{Y} &= \sum_{k\geq 0} \mathcal{Y}^k,\\ \mathcal{Y}^P &= \sum_{k\geq 0} \mathcal{Y}^{kP}.\end{aligned}$$

We thus have the following result.

*Proposition 4.1:* Suppose the nominal system $\Sigma$ is input–output decouplable by dynamic compensation. If

$$\mathcal{Y}_i^k = \mathcal{Y}_i^{kP}, \quad \text{for } 0 \leq k \leq n_{ie} - 1,\ 1 \leq i \leq p \tag{20}$$

and

$$\mathcal{X} \cap \mathcal{Y} = \mathcal{X} \cap \mathcal{Y}^P \tag{21}$$

then

i) the structures at infinity of system $\hat{\Sigma}_d^P$ and system $\hat{\Sigma}_d$ are the same, and

ii) the uncertain dynamics $\hat{\Sigma}_d^P$ becomes, in the new coordinates $\zeta = (\xi, \hat{\xi})$

$$\hat{\Sigma}_d^P: \begin{cases}\dot{\xi}_i = A_i\xi_i + b_iv_i + \psi_i(\xi, \hat{\xi}) + \tilde{F}_i(\xi, \hat{\xi}) + \tilde{G}_i(\xi, \hat{\xi})v,\\ \dot{\hat{\xi}} = \hat{f}(\xi, \hat{\xi}) + \hat{g}(\xi, \hat{\xi}, v),\\ y_i = \xi_{i1}, \quad i = 1, \cdots, p\end{cases} \tag{22}$$

with $A_i$ and $b_i$ in the canonical form (8), and $\psi_i$, $\tilde{F}_i$, and $\tilde{G}_i$ given by

$$\psi_i(\xi, \hat{\xi}) = \begin{bmatrix}\psi_{i1}(\xi_{i1}, \xi_{i2})\\ \vdots\\ \psi_{i,n_{ie}-1}(\xi_{i1}, \xi_{i2}, \cdots, \xi_{i,n_{ie}})\\ \psi_{i,n_{ie}}(\xi, \hat{\xi})\end{bmatrix},$$

$$\tilde{F}_i(\xi, \hat{\xi}) = \begin{bmatrix}0\\ \vdots\\ 0\\ \Delta_i(\xi, \hat{\xi})F(\xi, \hat{\xi})\end{bmatrix},$$

$$\tilde{G}_i(\xi, \hat{\xi}) = \begin{bmatrix}0\\ \vdots\\ 0\\ \Delta_i(\xi, \hat{\xi})G(\xi, \hat{\xi})\end{bmatrix}$$

where $\psi_{ij}(\xi, \hat{\xi}) = L_{\Delta f_e}L_{f_e}^{j-1}h_i(x_e)\ |_{x_e=\Phi^{-1}(\xi,\hat{\xi})}$ and $\Delta_i(\xi, \hat{\xi}) = L_{\Delta g_e}L_{f_e}^{n_{ie}-1}h_i(x_e)\ |_{x_e=\Phi^{-1}(\xi,\hat{\xi})}$, with $\Delta f_e(x_e) = \text{col}(\Delta f(x) + \Delta g(x)F(x, z), 0)$ and $\Delta g_e(x_e) = \text{col}(\Delta g(x)G(x, z), 0)$.

The conditions given in Proposition 4.1 are also necessary; these can be verified by standard computations. For the special case where $\psi_{ij} = 0$, for $1 \leq j \leq n_{ie} - 1$, the necessary and sufficient condition becomes

$$\Delta f + \text{span}\{\Delta g\} \perp \mathcal{X} \cap \mathcal{Y}.$$

*Proof of Proposition 4.1:* From the assumptions $\mathcal{X} \cap \mathcal{Y}^k = \mathcal{X} \cap \mathcal{Y}^{kP}$, for $k \geq 0$, thus

$$\begin{aligned}\dim \mathcal{X} \cap \mathcal{Y}^k &= \dim \mathcal{X} \cap \mathcal{Y}^{kP}\\ &= p + (p - \sigma_1) + \cdots + (p - \sigma_k)\end{aligned}$$

and the first statement follows. Since $\mathcal{Y}_i^1 = \mathcal{Y}_i^{1P}$, one has

$$\dot{\xi}_{i1} = \xi_{i2} + \psi_{i1}(\xi_{i1}, \xi_{i2})$$

where $\xi_{i2} = L_{f_e}h_i(x_e)$ and $\psi_{i1} = L_{\Delta f_e}h_i(x_e)$. In a similar vein, we have, for any $k = 2, \cdots, n_{ie} - 1$

$$\dot{\xi}_{ik} = \xi_{i,k+1} + \psi_{ik}(\xi_{i1}, \cdots, \xi_{i,k+1})$$

TABLE I
GEOMETRIC CONDITIONS IMPOSED ON THE NOMINAL AND UNCERTAIN NONLINEAR SYSTEMS

| Control Problem | Invariance Condition |
|---|---|
| Input-state linearization by static state feedback | $D_k = D_k^P$, for $k \geq 0$ |
| Input-output decoupling by static state feedback | $\mathcal{Y}_i^k = \mathcal{Y}_i^{kP}$, for $0 \leq k \leq r_i - 1$, $1 \leq i \leq p$, and $D^* = D^{*P}$ |
| Input-output decoupling by dynamic compensation | $\mathcal{Y}_i^k = \mathcal{Y}_i^{kP}$, for $0 \leq k \leq n_{ie} - 1$, $1 \leq i \leq p$, and $\mathcal{X} \cap \mathcal{Y} = \mathcal{X} \cap \mathcal{Y}^P$ |

with $\xi_{i,k+1} = L_{f_e}^k h_i(x_e)$ and $\upsilon_{ik} = L_{\Delta f_e} L_{f_e}^{k-1} h_i(x_e)$. Finally $\dot{\xi}_{i,n_{ie}}$ becomes

$$\dot{\xi}_{i,n_{ie}} = v_i + \upsilon_{i,n_{ie}}(\xi, \hat{\xi}) + \Delta_i(\xi, \hat{\xi}) F(\xi, \hat{\xi}) + \Delta_i(\xi, \hat{\xi}) G(\xi, \hat{\xi}) v$$

where, as well

$$\psi_{i,n_{ie}}(\xi, \hat{\xi}) = L_{\Delta f_e} L_{f_e}^{n_{ie}-1} h_i(x_e) \mid_{x_e = \Phi^{-1}(\xi, \hat{\xi})}$$

and

$$\Delta_i(\xi, \hat{\xi}) = L_{\Delta g_e} L_{f_e}^{n_{ie}-1} h_i(x_e) \mid_{x_e = \Phi^{-1}(\xi, \hat{\xi})}.$$

The unobservable dynamics $\dot{\hat{\xi}}$ takes the form

$$\dot{\hat{\xi}} = \hat{f}(\xi, \hat{\xi}) + \hat{g}(\xi, \hat{\xi}, v)$$

where now

$$\begin{aligned}\hat{f}(\xi, \hat{\xi}) = {} & L_f \hat{\xi}(x, z) \mid_{(x,z)=\Phi^{-1}(\xi,\hat{\xi})} \\ & + L_g \hat{\xi}.F(x, z) \mid_{(x,z)=\Phi^{-1}(\xi,\hat{\xi})} \\ & + L_M \hat{\xi}(x, z) \mid_{(x,z)=\Phi^{-1}(\xi,\hat{\xi})}\end{aligned}$$

$$\begin{aligned}\hat{g}(\xi, \hat{\xi}, v) = {} & L_g \hat{\xi}.G(x, z) \mid_{(x,z)=\Phi^{-1}(\xi,\hat{\xi})} \cdot v \\ & + L_{\Delta f} \hat{\xi}(x, z) \mid_{(x,z)=\Phi^{-1}(\xi,\hat{\xi})} \\ & + L_{\Delta g} \hat{\xi}.F(x, z) \mid_{(x,z)=\Phi^{-1}(\xi,\hat{\xi})} \\ & + L_{\Delta g} \hat{\xi}.G(x, z) \mid_{(x,z)=\Phi^{-1}(\xi,\hat{\xi})} \cdot v \\ & + L_N \hat{\xi}(x, z) \mid_{(x,z)=\Phi^{-1}(\xi,\hat{\xi})} \cdot v\end{aligned}$$

and representation (22) follows.

*Remark 4.2:* Note that the space $\mathcal{X} \cap \mathcal{Y}$ was first introduced in [2] for studying minimal dynamic compensation for input–output decoupling.

*Remark 4.3:* Proposition 4.1 applies to a most general situation, the dynamic input-state linearization problem which consists in the search of so-called linearizing outputs. Assume that the problem has a solution for $\Sigma$, then there exists linearizing outputs $\overline{y}$ and a dynamic compensator which decouples $\overline{y}$ and fully linearizes the closed-loop system [7]. If $\overline{\mathcal{Y}}_i^k$ and $\mathcal{X} \cap \overline{\mathcal{Y}}$ are invariant for $k \geq 0$, where $\overline{\mathcal{Y}}_i^k = \text{span}\{d\overline{y}_i, \cdots, d\overline{y}_i^{(k)}\}$ and $\overline{\mathcal{Y}} = \sum_{k \geq 0} \overline{\mathcal{Y}}^k$, then the uncertain closed-loop system has the form (22). It is unclear, however, if such conditions depend or not on the choice of the linearizing outputs; this remains an open problem.

The study of the effects of the uncertainties to the resultant closed-loop system when considering an output tracking problem is carried out in [3]. This is done by means of a Lyapunov-based approach recently reported in the literature [1], [11], [18].

## V. CONCLUSION

In this note, three control problems associated with uncertain MIMO nonlinear systems were treated: input-state linearization by static state feedback, input–output decoupling by static state feedback, and input–output decoupling by dynamic compensation. For each problem, we give geometric conditions which prove to describe intrinsic structure invariance properties of the nominal and uncertain systems; these conditions are summarized in Table I. A study of the effects of the uncertainties for which uniform boundedness of signals is retained for the closed-loop system when using dynamic compensation may be found in [3] as an application of the results in Section IV.

## REFERENCES


[1] S. Behtash, "Robust output tracking for nonlinear systems," *Int. J. Contr.*, vol. 51, pp. 1381–1407, 1990.
[2] L. Cao and Y. F. Zheng, "On minimal compensators for decoupling control," *Syst. Contr. Lett.*, pp. 121–128, 1992.
[3] R. Castro-Linares and C. Moog, "Robust output tracking by dynamic compensation," in *Proc. 1994 Amer. Contr. Conf.*, Baltimore, MD, 1994, pp. 2650–2694.
[4] M. Corless and G. Leitmann, "Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamical systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 1139–1144, 1981.
[5] M. D. Di Benedetto, J. W. Grizzle, and C. H. Moog, "Rank invariants of nonlinear systems," *SIAM J. Contr. Optim.*, vol. 27, pp. 658–672, 1989.
[6] M. Fliess, "Automatique et corps différentiels," *Forum Math.*, vol. 1, pp. 227–238, 1989.
[7] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "Sur les systèmes non linéaires différentiellement plats," *C.R.A.S. Paris*, vol. 315-I, pp. 619–624, 1992.
[8] L. C. Fu and T. L. Liao, "Globally stable tracking of nonlinear systems using variable structure control and with an application to a robotic manipulator," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1345–1351, 1990.
[9] A. Glumineau and C. H. Moog, "Essential orders and the nonlinear decoupling problem," *Int. J. Contr.*, vol. 50, pp. 1825–1834, 1989.
[10] I. J. Ha and E. G. Gilbert, "Robust tracking in nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 32, pp. 763–771, 1987.
[11] J. Hauser, S. S. Sastry, and G. Meyer, "Nonlinear controller design for flight control systems," Memo. No. UCB/ERL M88/76, Univ. California, 1988.
[12] H. J. C. Huijberts, H. Nijmeijer, and L. L. van der Wegen, "Minimality of dynamic input–output decoupling for nonlinear systems," *Syst. Contr. Lett.*, vol. 18, pp. 301–307, 1992.
[13] L. R. Hunt, R. Su, and G. Meyer, "Design for multi-input nonlinear systems," in *Differential Geometric Control Theory*, R. Brockett, R. Millmann, H. Sussmann (Eds.). pp. 268–298, 1983.
[14] A. Isidori, *Nonlinear Control Systems*. New York: Springer-Verlag, 1989.
[15] B. Jacubczyk and W. Respondek, "On linearization of control systems," *Bull. Acad. Pol. Sci. Math.*, vol. 18, pp. 517–522, 1980.
[16] H. Khalil, *Nonlinear Systems*. New York: McMillan, 1992.
[17] J. S. Liu and K. Yuan, "Characterization of allowable perturbations for robust decoupling of affine nonlinear systems," *Int. J. Contr.*, vol. 54, pp. 729–735, 1991.
[18] T. L. Liao, L. C. Fu, and C. F. Hsu, "Output tracking control of nonlinear systems with mismatched uncertainties," *Syst. Contr. Lett.*, vol. 18, pp. 39–47, 1992.
[19] R. Marino, "On the largest feedback linearizable system," *Syst. Contr. Lett.*, vol. 6, pp. 345–351, 1986.
[20] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*. New York: Springer-Verlag, 1990.
[21] S. Sastry and P. Kokotovic, "Feedback linearization in the presence of uncertainties," *Int. J. Adaptive Contr. Sig. Process.*, vol. 2, pp. 327–346, 1988.
[22] H. Sira-Ramirez, M. Zribi, and S. Ahmad, "Adaptive dynamical feedback regulation strategies for linearizable uncertain systems," *Int. J. Contr.*, vol. 57, pp. 121–139, 1993.
[23] J. J. E. Slotine and K. Hedrick, "Robust input–output feedback linearization," *Int. J. Contr.*, vol. 57, pp. 1133–1139, 1993.